On split Regular Hom-Leibniz algebras

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Abstract

We introduce the class of split regular Hom-Leibniz algebras as the natural generalization of split Leibniz algebras and split regular Hom-Lie algebras. By developing techniques of connections of roots for this kind of algebras, we show that such a split regular Hom-Leibniz algebra L is of the form $L = U + \sum\limits_{[j]\in\Lambda/\sim} I_{[j]}$ with U a subspace of the abelian subalgebra H and any $I_{[j]}$, a well described ideal of L, satisfying $[I_{[j]}, I_{[k]}] = 0$ if $[j] \neq [k]$. Under certain conditions, in the case of L being of maximal length, the simplicity of the algebra is characterized.

Key words: Hom-Leibniz algebra, Leibniz algebra, roots system, root space
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1 Introduction

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson and Silvestrov to describe the q-deformation of the Witt and the Virasoro algebras [1]. Since then, many authors have studied Hom-type algebras [2–7]. The notion of Leibniz algebras was introduced by Loday [8], which is a "nonantisymmetric" generalization of Lie algebras. So far, many results of this kind of algebras have been considered in the frameworks of low dimensional algebras, nilpotence and related problems [9–13]. In particular, Makhlouf and Silvestrov introduced the notion of Hom-Leibniz algebras in [14], which is a natural generalization of Leibniz algebras and Hom-Lie algebras.

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As is well-known, the class of the split algebras is specially related to addition quantum numbers, graded contractions, and deformations. For instance, for a physical system which displays a symmetry of L, it is interesting to know in detail the structure of the split decomposition because its roots can be seen as certain eigenvalues which are the additive quantum numbers characterizing the state of such system. Determining the structure of split algebras will become more and more meaningful in the area of research in mathematical physics. Recently, in [15–18], the structure of arbitrary split Lie algebras, arbitrary split Leibniz algebras, arbitrary split Lie triple systems and arbitrary split regular Hom-Lie algebras have been determined by the techniques of connections of roots. The purpose of this paper is to consider the structure of split regular Hom-Leibniz algebras by the techniques of connections of roots based on some work in [16, 17].

Throughout this paper, split regular Hom-Leibniz algebras L are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. This paper is organized as follows. In section 2, we establish the preliminaries on split regular Hom-Leibniz algebras theory. In section 3, we show that such an arbitrary regular Hom-Leibniz algebra L with a symmetric root system is of the form $L = U + \sum_{[j]\in\Lambda/\sim} I_{[j]}$ with U a subspace of the abelian subalgebra H and any $I_{[j]}$ a well described ideal of L, satisfying $[I_{[j]}, I_{[k]}] = 0$ if $[j] \neq [k]$. In section 4, we show that under certain conditions, in the case of L being of maximal length, the simplicity of the algebra is characterized.

2 Preliminaries

First we recall the definitions of Leibniz algebras, Hom-Lie algebras and Hom-Leibniz algebras.

Definition 2.1. [8] *A **right Leibniz algebra** L is a vector space over a base field $\mathbb{K}$ endowed with a bilinear product $[\cdot,\cdot]$ satisfying the Leibniz identity*

$$[[y,z],x] = [[y,x],z] + [y,[z,x]],$$

for all $x, y, z \in L$.

Definition 2.2. [5] *A **Hom-Lie algebra** L is a vector space over a base field $\mathbb{K}$ endowed with a bilinear product*

$$[\cdot,\cdot] : L \times L \to L$$

and with a linear map $\phi : L \to L$ such that

1. $[x,y] = -[y,x]$,
2. $[[x,y],\phi(z)] + [[y,z],\phi(x)] + [[z,x],\phi(y)] = 0$,

*for all $x, y, z \in L$. When ϕ furthermore is an algebra automorphism it is said that L is a **regular Hom-Lie algebra**.*

Definition 2.3. [14] *A **Hom-Leibniz algebra** L is a vector space over a base field $\mathbb{K}$ endowed with a bilinear product*

$$[\cdot,\cdot] : L \times L \to L$$

and with a linear map $\phi : L \to L$ satisfying the Hom-Leibniz identity

$$[[y,z],\phi(x)] = [[y,x],\phi(z)] + [\phi(y),[z,x]],$$

for all $x, y, z \in L$. When ϕ furthermore is an algebra automorphism it is said that L is a ***regular Hom-Leibniz algebra****.*

Clearly Hom-Lie algebras and Leibniz algebras are examples of Hom-Leibniz algebras.

Throughout this paper we will consider regular Hom-Leibniz algebras L being of arbitrary dimension and arbitrary base field $\mathbb{K}$. $\mathbb{N}$ denotes the set of all non-negative integers and $\mathbb{Z}$ denotes the set of all integers.

For any $x \in L$, we consider the adjoint mapping $\mathrm{ad}_x : L \to L$ defined by $\mathrm{ad}_x(z) = [z, x]$. A subalgebra A of L is a linear subspace such that $[A, A] \subset A$ and $\phi(A) = A$. A linear subspace I of L is called an ideal if $[I, L] + [L, I] \subset I$ and $\phi(I) = I$.

Let L be a Hom-Leibniz algebra, the ideal $J(L)$ generated by $\{[x, x] : x \in L\}$ plays an important role in the theory since it determines the non-Lie character of L. For convenience, write J for $J(L)$. From the Hom-Leibniz identity, this ideal satisfies

$$[L, J] = 0. \tag{2.1}$$

Let us introduce the class of split algebras in the framework of regular Hom-Leibniz algebras. Denote by H a maximal abelian subalgebra of a Hom-Leibniz algebra L. For a linear functional

$$\alpha : H \to \mathbb{K},$$

we define the root space of L (with respect to H) associated to α as the subspace

$$L_\alpha = \{v_\alpha \in L : [v_\alpha, h] = \alpha(h)\phi(v_\alpha) \; for \; any \; h \in H\}.$$

The elements $\alpha : H \to \mathbb{K}$ satisfying $L_\alpha \neq 0$ are called roots of L with respect to H. We denote $\Lambda := \{\alpha \in H^* \setminus \{0\} : L_\alpha \neq 0\}$.

Definition 2.4. *We say that L is a* ***split regular Hom-Leibniz algebra****, with respect to H, if*

$$L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha).$$

We also say that Λ is the roots system of L.

Note that when $\phi = \mathrm{Id}$, the split Leibniz algebras become examples of split regular Hom-Leibniz algebras. Hence, the present paper extends the results in [17]. For convenience, the mappings $\phi|_H$, $\phi|_H^{-1} : H \to H$ will be denoted by ϕ and ϕ^{-1} respectively.

Lemma 2.5. *For any α, $\beta \in \Lambda \cup \{0\}$, the following assertions hold.*

1. $\phi(L_\alpha) \subset L_{\alpha\phi^{-1}}$ *and* $\phi^{-1}(L_\alpha) \subset L_{\alpha\phi}$.
2. $[L_\alpha, L_\beta] \subset L_{\alpha\phi^{-1}+\beta\phi^{-1}}$.

Proof. 1. For $h \in H$ write $h' = \phi(h)$. Then for all $h \in H$ and $v_\alpha \in L_\alpha$, since $[v_\alpha, h] = \alpha(h)\phi(v_\alpha)$, one has

$$[\phi(v_\alpha), h'] = \phi([v_\alpha, h]) = \alpha(h)\phi(\phi(v_\alpha)) = \alpha\phi^{-1}(h')\phi(\phi(v_\alpha)).$$

Therefore we get $\phi(v_\alpha) \in L_{\alpha\phi^{-1}}$ and so $\phi(L_\alpha) \subset L_{\alpha\phi^{-1}}$. In a similar way, one gets $\phi^{-1}(L_\alpha) \subset L_{\alpha\phi}$.

2. For any $h \in H$, $v_\alpha \in L_\alpha$ and $v_\beta \in L_\beta$, by denoting $h' = \phi(h)$, by Hom-Leibniz identity, we have that

$$\begin{aligned}
[[v_\alpha, v_\beta], h'] = [[v_\alpha, v_\beta], \phi(h)] &= [[v_\alpha, h], \phi(v_\beta)] + [\phi(v_\alpha), [v_\beta, h]] \\
&= [\alpha(h)\phi(v_\alpha), \phi(v_\beta)] + [\phi(v_\alpha), \beta(h)\phi(v_\beta)] \\
&= (\alpha + \beta)(h)\phi([v_\alpha, v_\beta]) \\
&= (\alpha + \beta)\phi^{-1}(h')\phi([v_\alpha, v_\beta]).
\end{aligned}$$

Therefore we get $[v_\alpha, v_\beta] \in L_{\alpha\phi^{-1}+\beta\phi^{-1}}$ and so $[L_\alpha, L_\beta] \subset L_{\alpha\phi^{-1}+\beta\phi^{-1}}$. □

Lemma 2.6. *The following assertions hold.*

1. *If $\alpha \in \Lambda$ then $\alpha\phi^{-z} \in \Lambda$ for any $z \in \mathbb{Z}$.*
2. *$L_0 = H$.*

Proof. 1. It is a consequence of Lemma 2.5-1.

2. It is clear that the root space associated to the zero root satisfies $H \subset L_0$. Conversely, given any $v_0 \in L_0$ we can write

$$v_0 = h \oplus (\oplus_{i=1}^n v_{\alpha_i}),$$

where $h \in H$ and $v_{\alpha_i} \in L_{\alpha_i}$ for $i = 1, \cdots, n$, with $\alpha_i \neq \alpha_j$ if $i \neq j$. Hence

$$0 = [h \oplus (\oplus_{i=1}^n v_{\alpha_i}), h'] = \oplus_{i=1}^n \alpha_i(h')\phi(v_{\alpha_i}),$$

for any $h' \in H$. Hence Lemma 2.5-1 and the fact $\alpha_i \neq 0$ give us $v_{\alpha_i} = 0$ for $i = 1, \cdots, n$. So $v_0 = h \in H$. □

Definition 2.7. *A root system Λ of a split Hom-Leibniz algebra is called **symmetric** if it satisfies that $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$.*

3 Decompositions

In the following, L denotes a split regular Hom-Leibniz algebra with a symmetric root system Λ and $L = H \oplus (\oplus_{\alpha \in \Lambda} L_\alpha)$ the corresponding root decomposition. Given a linear functional $\alpha : H \to \mathbb{K}$, we denote by $-\alpha : H \to \mathbb{K}$ the element in H^* defined by $(-\alpha)(h) := -\alpha(h)$ for all $h \in H$. We begin by developing the techniques of connections of roots in this section.

Definition 3.1. *Let α and β be two nonzero roots. We shall say that α is* ***connected*** *to β if there exists $\alpha_1, \cdots, \alpha_k \in \Lambda$ such that*

If $k = 1$, then

1. $\alpha_1 \in \{a\phi^{-n} : n \in \mathbb{N}\} \cap \{\pm\beta\phi^{-m} : m \in \mathbb{N}\}$.

If $k \geq 2$, then

1. $\alpha_1 \in \{a\phi^{-n} : n \in \mathbb{N}\}$.
2. $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda$,
   $\alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1} \in \Lambda$,
   $\alpha_1\phi^{-3} + \alpha_2\phi^{-3} + \alpha_3\phi^{-2} + \alpha_4\phi^{-1} \in \Lambda$,
   $\cdots\cdots\cdots$
   $\alpha_1\phi^{-i} + \alpha_2\phi^{-i} + \alpha_3\phi^{-i+1} + \cdots + \alpha_{i+1}\phi^{-1} \in \Lambda$,
   $\cdots\cdots\cdots$
   $\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+2} + \alpha_3\phi^{-k+3} + \cdots + \alpha_i\phi^{-k+i} + \cdots + \alpha_{k-1}\phi^{-1} \in \Lambda$.
3. $\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_k\phi^{-1} \in \{\pm\beta\phi^{-m} : m \in \mathbb{N}\}$.

We shall also say that $\{\alpha_1, \cdots, \alpha_k\}$ is a connection from α to β.

Observe that the case $k = 1$ in Definition 3.1 is equivalent to the fact $\beta = \epsilon\alpha\phi^z$ for some $z \in \mathbb{Z}$ and $\epsilon \in \{\pm 1\}$.

By straightforward computations, we can easily get the following proposition:

Proposition 3.2. *Let L be a split Hom-Leibniz algebra relative to abelian subalgebra H with roots system Λ and J an ideal of L satisfying $[L, J] = 0$. Then the semi-direct product $\widehat{L} = L \rtimes L/J$ with respect to the operation given by*

$$[(a, x + J), (b, y + J)] = ([a, y] - [b, x], [x, y] + J)$$

is a Hom-Lie algebra which has weight decomposition relative to abelian subalgebra $H/J \cap H \subset L/J \subset \widehat{L}$, all adjoints are diagonalizable, and the roots system is exactly Λ :

$$\widehat{L}_\lambda = L_\lambda \oplus (L_\lambda / L_\lambda \cap J).$$

From Proposition 3.2, we can see roots system Λ in a Hom-Leibniz algebra is the same in a Hom-Lie algebra. By [16, Lemmas 2.2, 2.3, Proposition 2.4], we get the following Lemmas 3.3, 3.4 and Proposition 3.5.

Lemma 3.3. *For any $\alpha \in \Lambda$, we have that $\alpha\phi^{z_1}$ is connected to $\alpha\phi^{z_2}$ for every $z_1, z_2 \in \mathbb{Z}$. We also have that $\alpha\phi^{z_1}$ is connected to $-\alpha\phi^{z_2}$ in case $-\alpha\phi^{z_2} \in \Lambda$.*

Lemma 3.4. *Let $\{\alpha_1, \cdots, \alpha_k\}$ be a connection from α to β. Then the following assertions hold.*

1. *Suppose* $\alpha_1 = \alpha\phi^{-n}, n \in \mathbb{N}$. *Then for any* $r \in \mathbb{N}$ *such that* $r \geq n$, *there exists a connection* $\{\overline{\alpha}_1, \cdots, \overline{\alpha}_k\}$ *from* α *to* β *such that* $\overline{\alpha}_1 = \alpha\phi^{-r}$.

2. *Suppose that* $\alpha_1 = \epsilon\beta\phi^{-m}$ *in case* $k = 1$ *or*

$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_k\phi^{-1} = \epsilon\beta\phi^{-m}$$

in case $k \geq 2$, *with* $m \in \mathbb{N}$ *and* $\epsilon \in \{\pm 1\}$. *Then for any* $r \in \mathbb{N}$ *such that* $r \geq m$, *there exists a connection* $\{\overline{\alpha}_1, \cdots, \overline{\alpha}_k\}$ *from* α *to* β *such that* $\overline{\alpha}_1 = \epsilon\beta\phi^{-r}$ *in case* $k = 1$ *or*

$$\overline{\alpha}_1\phi^{-k+1} + \overline{\alpha}_2\phi^{-k+1} + \overline{\alpha}_3\phi^{-k+2} + \cdots + \overline{\alpha}_k\phi^{-1} = \epsilon\beta\phi^{-r}$$

in case $k \geq 2$.

Proposition 3.5. *The relation* $\sim$ *in* Λ, *defined by* $\alpha \sim \beta$ *if and only if* α *is connected to* β, *is of equivalence.*

Proposition 3.5 tells us the connection relation $\sim$ in Λ is an equivalence relation. So we denote by

$$\Lambda/\sim := \{[\alpha] : \alpha \in \Lambda\},$$

where $[\alpha]$ denotes the set of nonzero roots of L which are connected to α. Our next goal is to associate an adequate ideal $I_{[\alpha]}$ to any $[\alpha]$. For a fixed $\alpha \in \Lambda$, we define

$$I_{0,[\alpha]} := \mathrm{span}_{\mathbb{K}}\{[L_\beta, L_{-\beta}] : \beta \in [\alpha]\} \subset H$$

and

$$V_{[\alpha]} := \oplus_{\beta \in [\alpha]} L_\beta.$$

Then we denote by $I_{[\alpha]}$ the direct sum of the two subspaces above, that is,

$$I_{[\alpha]} := I_{0,[\alpha]} \oplus V_{[\alpha]}.$$

Proposition 3.6. *For any* $\alpha \in \Lambda$, *the linear subspace* $I_{[\alpha]}$ *is a subalgebra of* L.

Proof. First, it is sufficient to check that $I_{[\alpha]}$ satisfies $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$. By $I_{0,[\alpha]} \subset H$, it is clear that $[I_{0,[\alpha]}, I_{0,[\alpha]}]$=0 and we have

$$[I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\beta]} \oplus V_{[\beta]}] \subset [I_{0,[\alpha]}, V_{[\beta]}] + [V_{[\alpha]}, [I_{0,[\beta]}] + [V_{[\alpha]}, V_{[\beta]}]. \tag{3.2}$$

Let us consider the first summand in (3.2). For $\beta \in [\alpha]$, by Lemmas 2.5 and 3.3, one gets $[I_{0,[\alpha]}, L_\beta] \subset L_{\beta\phi^{-1}}$, where $\beta\phi^{-1} \in [\alpha]$. Hence

$$[I_{0,[\alpha]}, V_{[\alpha]}] \subset V_{[\alpha]}. \tag{3.3}$$

Similarly, we can also get

$$[V_{[\alpha]}, [I_{0,[\alpha]}] \subset V_{[\alpha]}. \tag{3.4}$$

Next, we consider the third summand in (3.2). Given β, $\gamma \in [\alpha]$ such that $[L_\beta, L_\gamma] \neq 0$, if $\gamma = -\beta$, we have $[L_\beta, L_\gamma] = [L_\beta, L_{-\beta}] \subset I_{0,[\alpha]}$. Suppose $\gamma \neq -\beta$, by Lemma 2.5-2, one

gets $\beta\phi^{-1} + \gamma\phi^{-1} \in \Lambda$. Therfore, we get $\{\beta, \gamma\}$ is a connection from β to $\beta\phi^{-1} + \gamma\phi^{-1}$. The transitivity of $\sim$ gives that $\beta\phi^{-1} + \gamma\phi^{-1} \in [\alpha]$ and so $[L_\beta, L_\gamma] \subset L_{\beta\phi^{-1}+\gamma\phi^{-1}} \subset V_{[\alpha]}$. Hence

$$[\oplus_{\beta\in[\alpha]} L_\beta, \oplus_{\beta\in[\alpha]} L_\beta] \subset I_{0,[\alpha]} \oplus V_{[\alpha]}.$$

That is,

$$[V_{[\alpha]}, V_{[\alpha]}] \subset I_{[\alpha]}. \tag{3.5}$$

From (3.2), (3.3), (3.4) and (3.5), we get $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$.

Second, we have to verify that $\phi(I_{[\alpha]}) = I_{[\alpha]}$. It is a direct consequence of Lemmas 2.5-1 and 3.3. □

Proposition 3.7. *If* $[\alpha] \neq [\beta]$*, then* $[I_{[\alpha]}, I_{[\beta]}] = 0$.

Proof. We have

$$[I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\beta]} \oplus V_{[\beta]}] \subset [I_{0,[\alpha]}, V_{[\beta]}] + [V_{[\alpha]}, [I_{0,[\beta]}] + [V_{[\alpha]}, V_{[\beta]}]. \tag{3.6}$$

Let us consider the third summand $[V_{[\alpha]}, V_{[\beta]}]$ in (3.6) and suppose there exist $\alpha_1 \in [\alpha]$ and $\alpha_2 \in [\beta]$ such that $[L_{\alpha_1}, L_{\alpha_2}] \neq 0$. By known condition $[\alpha] \neq [\beta]$, one gets $\alpha_1 \neq -\alpha_2$. So $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda$. Hence $\{\alpha_1, \alpha_2, -\alpha_1\phi^{-1}\}$ is a connection from α_1 to α_2. By the transitivity of the connection relation, we have $\alpha \in [\beta]$, a contradiction. Hence $[L_{\alpha_1}, L_{\alpha_2}] = 0$ and so

$$[V_{[\alpha]}, V_{[\beta]}] = 0. \tag{3.7}$$

Next we consider the first summand $[I_{0,[\alpha]}, V_{[\beta]}]$ in (3.6). Let us take $\alpha_1 \in [\alpha]$ and $\alpha_2 \in [\beta]$ and conclude that

$$\alpha_2([L_{\alpha_1}, L_{-\alpha_1}]) = 0. \tag{3.8}$$

Indeed, by applying Hom-Leibniz identity and (3.7), one gets

$$[\phi(L_{\alpha_2}), [L_{\alpha_1}, L_{-\alpha_1}]] = 0. \tag{3.9}$$

By ϕ is an algebra automorphism and (3.9), one gets

$$\phi[(L_{\alpha_2}), \phi^{-1}[L_{\alpha_1}, L_{-\alpha_1}]] = 0, \tag{3.10}$$

that is

$$[(L_{\alpha_2}), \phi^{-1}[L_{\alpha_1}, L_{-\alpha_1}]] = 0, \tag{3.11}$$

where $\phi^{-1}[L_{\alpha_1}, L_{-\alpha_1}] \subset H$. Hence (3.11) gives

$$\alpha_2\phi^{-1}([L_{\alpha_1}, L_{-\alpha_1}]) = 0, \tag{3.12}$$

for any $\alpha_1 \in [\alpha]$ and $\alpha_2 \in [\beta]$. By Lemma 2.5-1 and ϕ is an algebra automorphism, we get

$$\phi([L_{\alpha_1}, L_{-\alpha_1}]) \subset [L_{\alpha_1\phi^{-1}}, L_{-\alpha_1\phi^{-1}}],$$

that is

$$[L_{\alpha_1}, L_{-\alpha_1}] \subset \phi^{-1}([L_{\alpha_1\phi^{-1}}, L_{-\alpha_1\phi^{-1}}])$$

and by (3.12), one gets

$$\alpha_2([L_{\alpha_1}, L_{-\alpha_1}]) = 0.$$

From $[L_{\alpha_2}, [L_{\alpha_1}, L_{-\alpha_1}]] \subset \alpha_2([L_{\alpha_1}, L_{-\alpha_1}])\phi(L_{\alpha_2}) = 0$, we prove that $[I_{0,[\alpha]}, V_{[\beta]}] = 0$. In a similar way, we get $[V_{[\alpha]}, I_{0,[\beta]}] = 0$ and we conclude, together with (3.6) and (3.7), that $[I_{[\alpha]}, I_{[\beta]}] = 0$. □

Definition 3.8. *A Hom-Leibniz algebra L is said to be* ***simple*** *if its product is nonzero and its only ideals are $\{0\}$, J and L.*

It should be noted that the above definition agrees with the definition of simple Hom-Lie algebra, since $J = \{0\}$ in this case.

Theorem 3.9. *The following assertions hold.*

1. *For any $\alpha \in \Lambda$, the subalgebra*

$$I_{[\alpha]} = I_{0,[\alpha]} \oplus V_{[\alpha]}$$

of L associated to $[\alpha]$ is an ideal of L.

2. *If L is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$ and $H = \sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}]$.*

Proof. 1. Since $[I_{[\alpha]}, H] + [H, I_{[\alpha]}] = [I_{[\alpha]}, L_0] + [L_0, I_{[\alpha]}] \subset V_{[\alpha]}$, taking into account Propositions 3.6 and 3.7, we have

$$[I_{[\alpha]}, L] = [I_{[\alpha]}, H \oplus (\oplus_{\beta\in[\alpha]} L_\beta) \oplus (\oplus_{\gamma\notin[\alpha]} L_\gamma)] \subset I_{[\alpha]}$$

and

$$[L, I_{[\alpha]}] = [H \oplus (\oplus_{\beta\in[\alpha]} L_\beta) \oplus (\oplus_{\gamma\notin[\alpha]} L_\gamma), I_{[\alpha]}] \subset I_{[\alpha]}.$$

As we also have by Lemmas 2.5-1 and 3.3 that $\phi(I_{[\alpha]}) = I_{[\alpha]}$, we conclude that $I_{[\alpha]}$ is an ideal of I.

2. The simplicity of L implies $I_{[\alpha]} \in \{J, L\}$ for any $\alpha \in \Lambda$. If some $\alpha \in \Lambda$ is such that $I_{[\alpha]} = L$ then $[\alpha] = \Lambda$. Hence, L has all of its nonzero roots connected and $H = \sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}]$. Otherwise, if $I_{[\alpha]} = J$ for any $\alpha \in \Lambda$ then $[\alpha] = [\beta]$ for any $\alpha, \beta \in \Lambda$ and so $[\alpha] = \Lambda$. We also conclude that L has all of its nonzero roots connected and $H = \sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}]$. □

Theorem 3.10. *For a vector space complement U of $span_{\mathbb{K}}\{[L_\alpha, L_{-\alpha}] : \alpha \in \Lambda\}$ in H, we have*

$$L = U + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is one of the ideas of L described in Theorem 3.9-1, satisfying $[I_{[\alpha]}, I_{[\beta]}] = 0$, whenever $[\alpha] \neq [\beta]$.

Proof. Each $I_{[\alpha]}$ is well defined and, by Theorem 3.9-1, an ideal of L. It is clear that

$$L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha) = U + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}.$$

Finally Proposition 3.7 gives us $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. □

Definition 3.11. *The **annihilator** of a Hom-Leibniz algebra L is the set $\mathrm{Z}(L) = \{x \in L : [x, L] + [L, x] = 0\}$.*

Corollary 3.12. *If $Z(L) = 0$ and $[L, L] = L$, then L is the direct sum of the ideals given in Theorem 3.9,*

$$L = \oplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}.$$

Proof. From $[L, L] = L$, it is clear that $L = \oplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$. Finally, the sum is direct because $Z(L) = 0$ and $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. □

4 The simplicity of Split regular Hom-Leibniz algebras of maximal length.

In this section we focus on the simplicity of split regular Hom-Leibniz algebras by centering our attention in those of maximal length. From now on char($\mathbb{K}$)=0.

Definition 4.1. *We say that a split regular Hom-Leibniz algebra L is of **maximal length** if $\dim L_\alpha$=1 for any $\alpha \in \Lambda$.*

Lemma 4.2. *Let L be a split regular Hom-Leibniz algebra with $\mathrm{Z}(L) = 0$ and I an ideal of L. If $I \subset H$ then $I = \{0\}$.*

Proof. Suppose there exists a nonzero ideal I of L such that $I \subset H$. We get $[I, H] + [H, I] \subset [H, H] = 0$. We also get $[I, \oplus_{\alpha\in\Lambda} L_\alpha] + [\oplus_{\alpha\in\Lambda} L_\alpha, I] \subset I \subset H$. Then taking into account $H = L_0$, we have $[I, \oplus_{\alpha\in\Lambda} L_\alpha] + [\oplus_{\alpha\in\Lambda} L_\alpha, I] \subset H \cap (\oplus_{\alpha\in\Lambda} L_\alpha) = 0$. From here $I \subset \mathrm{Z}(L) = 0$, which is a contradiction. □

Lemma 4.3. *For any $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ there exists $h_0 \in H$ such that $\alpha(h_0) \neq 0$ and $\alpha(h_0) \neq \beta(h_0)$.*

Proof. This can be proved completely analogously to [16, Lemma 4.2]. □

Lemma 4.4. *Let $L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha)$ be a split regular Hom-Leibniz algebra. If I is an ideal of L then $I = (I \cap H) \oplus (\oplus_{\alpha\in\Lambda}(I \cap L_\alpha))$.*

Proof. Let $x \in I$. We can write $x = h + \sum_{j=1}^{n} v_{\alpha_j}$ with $h \in H$, $v_{\alpha_j} \in L_{\alpha_j}$ and $\alpha_j \neq \alpha_k$ if $j \neq k$. Let us show that any $v_{\alpha_j} \in I$.

If $n = 1$ we write $x = h + v_{\alpha_1} \in I$. By taking $h^{'} \in H$ such that $\alpha_1(h^{'}) \neq 0$ we have $[x, h^{'}] = \alpha_1(h^{'})\phi(v_{\alpha_1}) \in I$ and so $\phi(v_{\alpha_1}) \in I$. Therefore $\phi^{-1}(\phi(v_{\alpha_1})) = v_{\alpha_1} \in I$.

Suppose now $n > 1$ and consider α_1 and α_2. By Lemma 4.3 there exists $h_0 \in H$ such that $\alpha_1(h_0) \neq 0$ and $\alpha_1(h_0) \neq \alpha_2(h_0)$. Then we have

$$I \ni [x, h_0] = \alpha_1(h_0)\phi(v_{\alpha_1}) + \alpha_2(h_0)\phi(v_{\alpha_2}) + \cdots + \alpha_n(h_0)\phi(v_{\alpha_n}) \tag{4.13}$$

and

$$I \ni \phi(x) = \phi(h) + \phi(v_{\alpha_1}) + \phi(v_{\alpha_2}) + \cdots + \phi(v_{\alpha_n}). \tag{4.14}$$

By multiplying (4.14) by $\alpha_2(h_0)$ and subtracting (4.13), one gets

$$\begin{aligned} &\alpha_2(h_0)\phi(h) + (\alpha_2(h_0) - \alpha_1(h_0))\phi(v_{\alpha_1}) + (\alpha_2(h_0) - \alpha_3(h_0))\phi(v_{\alpha_3}) \\ &+ \cdots + (\alpha_2(h_0) - \alpha_n(h_0))\phi(v_{\alpha_n}) \in I. \end{aligned}$$

By denoting $\widetilde{h} := \alpha_2(h_0)\phi(h) \in H$ and $v_{\alpha_i\phi^{-1}} := (\alpha_2(h_0) - \alpha_i(h_0))\phi(v_{\alpha_i}) \in L_{\alpha_i\phi^{-1}}$, we can write

$$\widetilde{h} + v_{\alpha_1\phi^{-1}} + v_{\alpha_3\phi^{-1}} + \cdots + v_{\alpha_n\phi^{-1}} \in I. \tag{4.15}$$

Now we can argue as above, with (4.15), to get

$$\widetilde{\widetilde{h}} + v_{\alpha_1\phi^{-2}} + v_{\alpha_4\phi^{-2}} + \cdots + v_{\alpha_n\phi^{-2}} \in I,$$

for $\widetilde{\widetilde{h}} \in H$ and $v_{\alpha_1\phi^{-2}} \in L_{\alpha_i\phi^{-2}}$. Following this process, we obtain

$$\overline{h} + v_{\alpha_1\phi^{-n+1}} \in I,$$

with $\overline{h} \in H$ and $v_{\alpha_1\phi^{-n+1}} \in L_{\alpha_i\phi^{-n+1}}$. As in the above case $n = 1$, we conclude that $v_{\alpha_1\phi^{-n+1}} \in I$ and consequently $v_{\alpha_1} = \phi^{-n+1}(v_{\alpha_1\phi^{-n+1}}) \in I$.

In a similar way we can prove that $v_{\alpha_i} \in I$ for $i \in \{2, \cdots, n\}$, and the proof is complete. □

Let us return to a split regular Hom-Leibniz algebra of maximal length L. From now on $L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha)$ denotes a split Hom-Leibniz algebra of maximal length. By Lemma 4.4, we assert that given any nonzero ideal I of L then

$$I = (I \cap H) \oplus (\oplus_{\alpha\in\Lambda^I} L_\alpha), \tag{4.16}$$

where $\Lambda^I := \{\alpha \in \Lambda : I \cap L_\alpha \neq 0\}$.

In particular, case I=J, we get

$$J = (J \cap H) \oplus (\oplus_{\alpha\in\Lambda^J} L_\alpha). \tag{4.17}$$

From here, we can write

$$\Lambda = \Lambda^J \cup \Lambda^{\neg J}, \tag{4.18}$$

where

$$\Lambda^J := \{\alpha \in \Lambda : L_\alpha \subset J\}$$

and

$$\Lambda^{\neg J} := \{\alpha \in \Lambda : L_\alpha \cap J = 0\}.$$

Therefore

$$L = H \oplus (\oplus_{\alpha\in\Lambda^{\neg J}} L_\alpha) \oplus (\oplus_{\beta\in\Lambda^J} L_\beta). \tag{4.19}$$

We note that the fact that $L = [L, L]$, the split decomposition given by (4.19) and (2.1) show

$$H = \sum_{\alpha\in\Lambda^{\neg J}} [L_\alpha, L_{-\alpha}]. \tag{4.20}$$

Now, observe that the concept of connectivity of nonzero roots given in Definition 3.1 is not strong enough to detect if a given $\alpha \in \Lambda$ belongs to Λ^J or to $\Lambda^{\neg J}$. Consequently we lose the information respect to whether a given root space L_α is contained in J or not, which is fundamental to study the simplicity of L. So, we are going to refine the concept of connections of nonzero roots in the setup of maximal length split regular Hom-Leibniz algebras. The symmetry of Λ^J and $\Lambda^{\neg J}$ will be understood as usual. That is, Λ^γ,$\gamma \in \{J, \neg J\}$, is called symmetric if $\alpha \in \Lambda^\gamma$ implies $-\alpha \in \Lambda^\gamma$.

Definition 4.5. *Let $\alpha, \beta \in \Lambda^\gamma$ with $\gamma \in \{J, \neg J\}$. We say that α is $\neg J$-**connected** to β, denoted by $\alpha \sim_{\neg J} \beta$, if there exist*

$$\alpha_2, \cdots, \alpha_k \in \Lambda^{\neg J}$$

such that

1. $\{\alpha_1, \alpha_1\phi^{-1} + \alpha_2\phi^{-1}, \alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1}, \cdots, \alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_k\phi^{-1}\} \subset \Lambda^\gamma$,
2. $\alpha_1 \in \alpha\phi^{-n}$, *for $n \in \mathbb{N}$,*
3. $\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_k\phi^{-1} \in \pm\beta\phi^{-m}$, *for $m \in \mathbb{N}$.*

We shall also say that $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ is a $\neg J$-connection from α to β.

Proposition 4.6. *The following assertions hold.*

1. *If $\Lambda^{\neg J}$ is symmetric, then the relation $\sim_{\neg J}$ is an equivalence relation in $\Lambda^{\neg J}$.*

2. *If $L = [L, L]$ and $\Lambda^{\neg J}$, Λ^J are symmetric, then the relation $\sim_{\neg J}$ is an equivalence relation in Λ^J.*

Proof. 1. Can be proved in a similar way to Proposition 3.5.

2. Let $\beta \in \Lambda^J$. Since $\beta \neq 0$, (4.20) gives us that there exists $\alpha \in \Lambda^{\neg J}$ such that $[\phi(L_\beta), [L_\alpha, L_{-\alpha}]] \neq 0$. By Hom-Leibniz identity, either $[[L_\beta, L_\alpha], \phi(L_{-\alpha})] \neq 0$ or $[[L_\beta, L_{-\alpha}], \phi(L_\alpha)] \neq 0$. In the first case, the $\neg J$-connection $\{\beta, \alpha, -\alpha\phi^{-1}\}$ gives us $\beta \sim_{\neg J} \beta$ while in the second one the $\neg J$-connection $\{\beta, -\alpha, \alpha\phi^{-1}\}$ gives us the same result. Consequently $\sim_{\neg J}$ is reflexive in Λ^J. The symmetric and transitive character of $\sim_{\neg J}$ in Λ^J follows as in Proposition 3.5. ☐

Let us introduce the notion of root-multiplicativity in the framework of split regular Hom-Leibniz algebras of maximal length, in a similar way to the ones for split regular Hom-Lie algebras (see [16]).

Definition 4.7. *We say that a split regular Hom-Leibniz algebra of maximal length L is* ***root-multiplicative*** *if the below conditions hold.*

1. *Given $\alpha, \beta \in \Lambda^{\neg J}$ such that $\alpha\phi^{-1} + \beta\phi^{-1} \in \Lambda$ then $[L_\alpha, L_\beta] \neq 0$.*
2. *Given $\alpha \in \Lambda^{\neg J}$ and $\gamma \in \Lambda^J$ such that $\alpha\phi^{-1} + \gamma\phi^{-1} \in \Lambda^J$ then $[L_\alpha, L_\gamma] \neq 0$.*

Another interesting notion related to split regular Hom-Leibniz algebras of maximal length L is those of Lie-annihilator. Write $L = H \oplus (\oplus_{\alpha \in \Lambda^{\neg J}} L_\alpha) \oplus (\oplus_{\beta \in \Lambda^J} L_\beta)$ (see (4.19)).

Definition 4.8. *The* ***Lie-annihilator*** *of a split Hom-Leibniz algebra of maximal length L is the set*

$$\mathrm{Z}_{\mathrm{Lie}}(L) = \Big\{ x \in L : [x, H \oplus (\oplus_{\alpha \in \Lambda^{\neg J}} L_\alpha)] + [H \oplus (\oplus_{\alpha \in \Lambda^{\neg J}} L_\alpha), x] = 0 \Big\}.$$

Clearly we have $\mathrm{Z}(L) \subset \mathrm{Z}_{\mathrm{Lie}}(L)$.

Proposition 4.9. *Suppose $L = [L, L]$ and L is root-multiplicative. If $\Lambda^{\neg J}$ has all of its roots $\neg J$-connected, then any ideal I of L such that $I \not\subseteq H \oplus J$ satisfies $I = L$.*

Proof. By (4.16) and (4.18), we can write

$$I = (I \cap H) \oplus (\oplus_{\alpha_i \in \Lambda^{\neg J, I}} L_{\alpha_i}) \oplus (\oplus_{\beta_j \in \Lambda^{J,I}} L_{\beta_j}),$$

where $\Lambda^{\neg J,I} := \Lambda^{\neg J} \cap \Lambda^I$ and $\Lambda^{J,I} := \Lambda^J \cap \Lambda^I$. Since $I \not\subseteq H \oplus J$, one gets $\Lambda^{\neg J,I} \neq \emptyset$ and so we can take some $\alpha_0 \in \Lambda^{\neg J,I}$ such that

$$L_{\alpha_0} \subset I. \tag{4.21}$$

By Lemma 2.5-1, $\phi(L_{\alpha_0}) \subset L_{\alpha_0\phi^{-1}}$. Since L is of maximal length, we have $0 \neq \phi(L_{\alpha_0}) = L_{\alpha_0\phi^{-1}}$. (4.21) and ϕ is injective give us $\phi(L_{\alpha_0}) \subset \phi(I) = I$. So, $L_{\alpha_0\phi^{-1}} \subset I$. Similarly we get

$$L_{\alpha_0\phi^{-n}} \subset I, \text{ for } n \in \mathbb{N}. \tag{4.22}$$

For any $\beta \in \Lambda^{\neg J}$, $\beta \notin \pm\alpha_0\phi^{-n}$, for $n \in \mathbb{N}$, the fact that α_0 and β are $\neg J$-connected gives us a $\neg J$-connection $\{\gamma_1, \cdots, \gamma_k\} \subset \Lambda^{\neg J}$ from α_0 to β such that

1. $\{\gamma_1, \gamma_1\phi^{-1}+\gamma_2\phi^{-1}, \gamma_1\phi^{-2}+\gamma_2\phi^{-2}+\gamma_3\phi^{-1}, \cdots, \gamma_1\phi^{-k+1}+\gamma_2\phi^{-k+1}+\gamma_3\phi^{-k+2}+\cdots+\gamma_k\phi^{-1}\} \subset \Lambda^{\neg J}$,
2. $\gamma_1 \in \alpha_0\phi^{-n}$, for $n \in \mathbb{N}$,
3. $\gamma_1\phi^{-k+1}+\gamma_2\phi^{-k+1}+\gamma_3\phi^{-k+2}+\cdots+\gamma_k\phi^{-1} \in \pm\beta\phi^{-m}$, for $m \in \mathbb{N}$.

Consider γ_1, γ_2 and $\gamma_1\phi^{-1}+\gamma_2\phi^{-1}$. Since $\gamma_1, \gamma_2 \in \Lambda^{\neg J}$, the root-multiplicativity and maximal length of L show $[L_{\gamma_1}, L_{\gamma_2}] = L_{\gamma_1\phi^{-1}+\gamma_2\phi^{-1}}$, and by (4.22), $L_{\gamma_1} \subset I$. So we have

$$L_{\gamma_1\phi^{-1}+\gamma_2\phi^{-1}} \subset I.$$

We can argue in a similar way from $\gamma_1\phi^{-1}+\gamma_2\phi^{-1}, \gamma_3$, and $\gamma_1\phi^{-2}+\gamma_2\phi^{-2}+\gamma_3\phi^{-1}$ to get

$$L_{\gamma_1\phi^{-2}+\gamma_2\phi^{-2}+\gamma_3\phi^{-1}} \subset I.$$

Following this process with the $\neg J$-connection $\{\gamma_1, \cdots, \gamma_k\}$, we obtain that

$$L_{\gamma_1\phi^{-k+1}+\gamma_2\phi^{-k+1}+\gamma_3\phi^{-k+2}+\cdots+\gamma_k\phi^{-1}} \subset I.$$

From here, we get that either

$$L_{\beta\phi^{-m}} \subset I \text{ or } L_{-\beta\phi^{-m}} \subset I, \tag{4.23}$$

for any $\beta \in \Lambda^{\neg J}$, $m \in \mathbb{N}$. Note that $\beta \in \Lambda^{\neg J}$ gives us

$$\beta\phi^{-m} \in \Lambda^{\neg J}, \text{ for } m \in \mathbb{N}. \tag{4.24}$$

Since $H = \sum_{\beta\in\Lambda^{\neg J}}[L_\beta, L_{-\beta}]$, by (4.23) and (4.24), we get

$$H \subset I. \tag{4.25}$$

Now, given any $\delta \in \Lambda$, the facts $\delta \neq 0$, $H \subset I$ and the maximal length of L show that

$$[L_\delta, H] = L_\delta \subset I. \tag{4.26}$$

From (4.25) and (4.26), we conclude $I = L$. □

Proposition 4.10. *Suppose $L = [L, L]$, $\mathrm{Z}(L) = 0$ and L is root-multiplicative. If $\Lambda^{\neg J}$, Λ^J are symmetric and Λ^J has all of its roots $\neg J$-connected, then any nonzero ideal I of L such that $I \subseteq J$ satisfies either $I = J$ or $J = I \oplus K$ with K an ideal of L.*

Proof. By (4.16) and (4.18), we can write

$$I = (I \cap H) \oplus (\oplus_{\alpha_i\in\Lambda^{J,I}} L_{\alpha_i}),$$

where $\Lambda^{J,I} \subset \Lambda^J$. Observe that the fact $\mathrm{Z}(L) = 0$ implies

$$J \cap H = \{0\}. \tag{4.27}$$

Indeed, we have $[L_\alpha, J\cap H]+[J\cap H, L_\alpha]\subset [L,J]=0$, for any $\alpha\in\Lambda^J$ and $[H, J\cap H]+[J\cap H, H]=0$. So, $[L, J\cap H]+[J\cap H, L]=0$. That is, $J\cap H\subset \mathrm{Z}(L)=0$. Hence we can write

$$I=\oplus_{\alpha_i\in\Lambda^{J,I}}L_{\alpha_i},$$

with $\Lambda^{J,I}\neq\emptyset$, and so we can take some $\alpha_0\in\Lambda^{J,I}$ such that $L_{\alpha_0}\subset I$. We can argue with the root-multiplicativity and the maximal length of L as in Proposition 4.9 to conclude that given any $\beta\in\Lambda^J$, there exists a $\neg J$-connection $\{\gamma_1,\cdots,\gamma_k\}$ from α_0 to β such that

$$[[\cdots[L_{\gamma_1},L_{\gamma_2}],\cdots],L_{\gamma_k}]\in L_{\pm\beta\phi^{-m}},\quad \text{for}\quad m\in\mathbb{N}$$

and so

$$L_{\epsilon\beta\phi^{-m}}\subset I,\quad \text{for}\ \ \text{some}\ \epsilon\in\pm1,\quad m\in\mathbb{N}. \tag{4.28}$$

Note that $\beta\in\Lambda^J$ indicates $L_\beta\subset J$. By ϕ is injective, we get $\phi(L_\beta)\subset\phi(J)=J$. By Lemma 2.5-1, $\phi(L_\beta)\subset L_{\beta\phi^{-1}}$. Since L is of maximal length, we have $0\neq\phi(L_\beta)=L_{\beta\phi^{-1}}$. So, $L_{\beta\phi^{-1}}\subset J$. Similarly we get

$$L_{\beta\phi^{-m}}\subset J,\ \text{for}\ m\in\mathbb{N}. \tag{4.29}$$

By (4.28) and (4.29), we easily get

$$\epsilon_\beta\beta\phi^{-m}\in\Lambda^{J,I}\ \text{for any}\ \beta\in\Lambda^J,\quad \text{some}\ \epsilon_\beta\in\pm1\ \ \text{and}\ \ m\in\mathbb{N}. \tag{4.30}$$

Suppose $-\alpha_0\in\Lambda^{J,I}$. Then we also have that $\{-\gamma_1,\cdots,-\gamma_k\}$ from $-\alpha_0$ to β is a $\neg J$-connection from $-\alpha_0$ to β satisfying

$$[[\cdots[L_{-\gamma_1},L_{-\gamma_2}],\cdots],L_{-\gamma_k}]\in L_{-\epsilon_\beta\beta\phi^{-m}}\subset I$$

and so $L_{\beta\phi^{-m}}+L_{-\beta\phi^{-m}}\subset I$. Hence, (4.17) and (4.27) imply that $I=J$.

Now, suppose there is not any $\alpha_0\in\Lambda^{J,I}$ such that $-\alpha_0\in\Lambda^{J,I}$. (4.30) allows us to write $\Lambda^J=\Lambda^{J,I}\cup(-\Lambda^{J,I})$. By denoting $K=\oplus_{\alpha_i\in\Lambda^{J,I}}L_{-\alpha_i}$, we have

$$J=I\oplus K. \tag{4.31}$$

Let us finally show that K is an ideal of L. We have $[L,K]\subset[L,J]=0$ and

$$[K,L]\subset[K,H]+[K,\oplus_{\beta\in\Lambda^{\neg J}}L_\beta]+[K,\oplus_{\gamma\in\Lambda^J}L_\gamma]\subset K+[K,\oplus_{\beta\in\Lambda^{\neg J}}L_\beta].$$

Let us consider the last summand $[K,\oplus_{\beta\in\Lambda^{\neg J}}L_\beta]$ and suppose there exist $\alpha_i\in\Lambda^{J,I}$ and $\beta\in\Lambda^{\neg J}$ such that $[L_{-\alpha_i},L_\beta]\neq0$. Since $L_{-\alpha_i}\subset K\subset J$, we get $-\alpha_i\phi^{-1}+\beta\phi^{-1}\in\Lambda^J$. By the root-multiplicativity of L, the symmetries of Λ^J and $\Lambda^{\neg J}$, and the fact $L_{\alpha_i}\subset I$, one gets $0\neq[L_{\alpha_i},L_{-\beta}]=L_{\alpha_i\phi^{-1}-\beta\phi^{-1}}\subset I$, that is, $\alpha_i\phi^{-1}-\beta\phi^{-1}\in\Lambda^{J,I}$. Hence, $-\alpha_i\phi^{-1}+\beta\phi^{-1}\in-\Lambda^{J,I}$ and so $[L_{-\alpha_i},L_\beta]\subset K$. Consequently $[K,\oplus_{\beta\in\Lambda^{\neg J}}L_\beta]\subset K$.

Next, we have to verify that $\phi(K)=K$. Indeed, since I, J are two nonzero ideals, we have $\phi(I)=I$ and $\phi(J)=J$. By (4.31) and ϕ is an algebra automorphism, it is easily to get $\phi(K)=K$. We conclude that K is an ideal of L. $\square$

We introduce the definition of primeness in the framework of Hom-Leibniz algebras following the same motivation that in the case of simplicity (see Definition 3.8 and the above paragraph).

Definition 4.11. *A Hom-Leibniz algebra L is said to be **prime** if given two ideals I, K of L satisfying $[I,K]+[K,I]=0$, then either $I \in \{0,J,L\}$ or $K \in \{0,J,L\}$.*

We also note that the above definition agrees with the definition of prime Hom-Lie algebra, since $J=0$ in this case.

Under the hypotheses of Proposition 4.10 we have:

Corollary 4.12. *If furthermore L is prime, then any nonzero ideal I of L such that $I \subseteq J$ satisfies $I = J$.*

Proof. Observe that, by Proposition 4.10, we could have $J = I \oplus K$ with I, K ideals of L, being $[I,K]+[K,I]=0$ as consequence of $I, K \subseteq J$. The primeness of L completes the proof. □

Proposition 4.13. *Suppose $L=[L,L]$, $\mathrm{Z}_{\mathrm{Lie}}(L)=0$ and L is root-multiplicative. If $\Lambda^{\neg J}$ has all of its roots $\neg J$-connected, then any ideal I of L such that $I \not\subseteq J$ satisfies $I = L$.*

Proof. Taking into account Lemma 4.4 and Proposition 4.9 we just have to study the case in which

$$I = (I \cap H) \oplus (\oplus_{\beta_j \in \Lambda^{J,I}} L_{\beta_j}),$$

where $I \cap H \neq 0$. But this possibility never happens. Indeed, observe that $[L_\alpha, I \cap H] + [I \cap H, L_\alpha] \subset [L_\alpha, H] + [H, L_\alpha] \subset L_\alpha$, for any $\alpha \in \Lambda^{\neg J}$ and $[L_\alpha, I \cap H] + [I \cap H, L_\alpha] \subset [L_\alpha, I] + [I, L_\alpha] \subset I$. So, $[L_\alpha, I \cap H] + [I \cap H, L_\alpha] \subset L_\alpha \cap I = 0$, for $\alpha \in \Lambda^{\neg J}$. Since we also have $[I \cap H, H] + [H, I \cap H] \subset [H,H] = 0$, one gets $I \cap H \subset \mathrm{Z}_{\mathrm{Lie}}(L) = 0$, a contradiction. Proposition 4.9 completes the proof. □

Given any $\alpha \in \Lambda^\gamma$, $\gamma \in \{J, \neg J\}$, we denote by

$$\Lambda^\gamma_\alpha := \{\beta \in \Lambda^\gamma : \beta \sim_{\neg J} \alpha\}.$$

For $\alpha \in \Lambda^\gamma$, we write $H_{\Lambda^\gamma_\alpha} := \mathrm{span}_{\mathbb{K}}\{[L_\beta, L_{-\beta}] : \beta \in \Lambda^\gamma_\alpha\} \subset H$, and $V_{\Lambda^\gamma_\alpha} := \oplus_{\beta \in \Lambda^\gamma_\alpha} L_\beta$. We denote by $L_{\Lambda^\gamma_\alpha}$ the following subspace of L, $L_{\Lambda^\gamma_\alpha} := H_{\Lambda^\gamma_\alpha} \oplus V_{\Lambda^\gamma_\alpha}$.

Lemma 4.14. *If $L=[L,L]$, then $L_{\Lambda^J_\alpha}$ is an ideal of L for any $\alpha \in \Lambda^J$.*

Proof. By (4.20), we get $H_{\Lambda^J_\alpha} = 0$ and so

$$L_{\Lambda^J_\alpha} = \oplus_{\beta \in \Lambda^J_\alpha} L_\beta.$$

It is easy to see that

$$[L_\delta, L_{\Lambda^J_\alpha}] + [L_{\Lambda^J_\alpha}, L_\delta] \subset [L, J] = 0, \quad \text{for} \quad \delta \in \Lambda^J \tag{4.32}$$

and

$$[L_{\Lambda^J_\alpha}, H] + [H, L_{\Lambda^J_\alpha}] \subset L_{\Lambda^J_\alpha}. \tag{4.33}$$

We will show that

$$[L_{\Lambda^J_\alpha}, L_\gamma] \subset L_{\Lambda^J_\alpha}, \quad \text{for any} \quad \gamma \in \Lambda^{\neg J}. \tag{4.34}$$

Indeed, given any $\beta \in \Lambda^J_\alpha$ such that $[L_\beta, L_\gamma] \neq 0$ we have $\beta\phi^{-1} + \gamma\phi^{-1} \in \Lambda^J$ and so $\{\beta, \gamma\}$ is a $\neg J$-connection from β to $\beta\phi^{-1} + \gamma\phi^{-1}$. By the symmetry and transitivity of $\sim_{\neg J}$ in Λ^J, we get $\beta\phi^{-1} + \gamma\phi^{-1} \in \Lambda^J_\alpha$. Hence $[L_\beta, L_\gamma] \subset L_{\Lambda^J_\alpha}$, that is, (4.34) holds. taking into account (4.19), by (4.32), (4.33) and (4.34), we have

$$[L, L_{\Lambda^J_\alpha}] + [L_{\Lambda^J_\alpha}, L] \subset L_{\Lambda^J_\alpha}. \tag{4.35}$$

Next, we have to verify that

$$\phi(L_{\Lambda^J_\alpha}) = L_{\Lambda^J_\alpha}. \tag{4.36}$$

Indeed, given $\beta \in \Lambda^J_\alpha$ such that $L_\beta \subset L_{\Lambda^J_\alpha}$ and we have $[L_\beta, J] = 0$. By ϕ is an algebra automorphism, one gets

$$[\phi(L_\beta), \phi(J)] = [\phi(L_\beta), J] = 0. \tag{4.37}$$

By Lemma 2.5-1 and L is of maximal length, we get $0 \neq \phi(L_\beta) = L_{\beta\phi^{-1}}$. Therefore, (4.37) gives us $L_{\beta\phi^{-1}} \in \Lambda^J$. That is, $\phi(L_\beta) \subset L_\tau$, for $\tau \in \Lambda^J$. Since L is of maximal length, $\phi(L_\beta) = L_\tau$, for $\tau \in \Lambda^J$. Hence (4.36) holds. Consequently, it follows from (4.35) and (4.36) that $L_{\Lambda^J_\alpha}$ is an ideal of L. $\square$

Theorem 4.15. *Suppose $L = [L, L]$, $Z_{\mathrm{Lie}}(L) = 0$, L is root-multiplicative. If Λ^J, $\Lambda^{\neg J}$ are symmetric then L is simple if and only if it is prime and Λ^J, $\Lambda^{\neg J}$ have all of their roots $\neg J$-connected.*

Proof. Suppose L is simple. If $\Lambda^J \neq \emptyset$ and we take $\alpha \in \Lambda^J$. Lemma 4.14 gives us $L_{\Lambda^J_\alpha}$ is a nonzero ideal of L. By L is simple, one gets $L_{\Lambda^J_\alpha} = J = \oplus_{\beta \in \Lambda^J} L_\beta$ (see (4.17) and (4.27)). Hence, $\Lambda^J_\alpha = \Lambda^J$ and consequently Λ^J has all of its roots $\neg J$-connected.

Consider now any $\gamma \in \Lambda^{\neg J}$ and the subspace $L_{\Lambda^{\neg J}_\gamma}$. Let us denote by $I(L_{\Lambda^{\neg J}_\gamma})$ the ideal of L generated by $L_{\Lambda^{\neg J}_\gamma}$. We observe that the fact J is an ideal of L and we assert that $I(L_{\Lambda^{\neg J}_\gamma}) \cap (\oplus_{\delta \in \Lambda^{\neg J}} L_\delta)$ is contained in the linear span of the set

$$\begin{aligned}
&\{[[\cdots[v_{\gamma'}, v_{\alpha_1}], \cdots], v_{\alpha_n}]; \quad [v_{\alpha_n}, [\cdots[v_{\alpha_1}, v_{\gamma'}], \cdots]];\\
&[[\cdots[v_{\alpha_1}, v_{\gamma'}], \cdots], v_{\alpha_n}]; \quad [v_{\alpha_n}, [\cdots[v_{\gamma'}, v_{\alpha_1}], \cdots]];\\
&\text{where } 0 \neq v_{\gamma'} \in L_{\Lambda^{\neg J}_\gamma},\ 0 \neq v_{\alpha_i} \in L_{\alpha_i},\ \alpha_i \in \Lambda^{\neg J} \text{and } n \in \mathbb{N}\}.
\end{aligned}$$

By simplicity $I(L_{\Lambda^{\neg J}_\gamma}) = L$. From here, given any $\delta \in \Lambda^{\neg J}$, the above observation gives us that we can write $\delta = \gamma'\phi^{-m} + \alpha_1\phi^{-m} + \alpha_2\phi^{-m+1} + \cdots + \alpha_m\phi^{-1}$ for $\gamma' \in \Lambda^{\neg J}_\gamma$, $\alpha_i \in \Lambda^{\neg J}$, $m \in \mathbb{N}$ and being the partial sums nonzero. Hence $\{\gamma', \alpha_1, \cdots, \alpha_m\}$ is a $\neg J$-connection from γ' to δ. By the symmetry and transitivity of $\sim_{\neg J}$ in $\Lambda^{\neg J}$, we deduce γ is $\neg J$-connected to any $\delta \in \Lambda^{\neg J}$. Consequently, Proposition 4.6 gives us that $\Lambda^{\neg J}$ has all of its roots $\neg J$-connected. Finally, since L is simple then is prime.

The converse is a consequence of Corollary 4.12 and Proposition 4.13. $\square$

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